

Symplectic involutions on deformations of $K3^{[2]}$

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Abstract

Let X be a Hyperkähler manifold deformation equivalent to the Hilbert square of a K3 surface and let φ be an involution preserving the symplectic form. We prove that the fixed locus of φ consists of 28 isolated points and 1 K3 surface, moreover the anti-invariant lattice of the induced involution on $H^2(X, \mathbb{Z})$ is isomorphic to $E_8(-2)$. Finally we prove that any couple consisting of one such manifold and a symplectic involution on it can be deformed into a couple consisting of the Hilbert square of a K3 surface and the involution induced by a symplectic involution on the K3 surface.

1 Introduction

The aim of the present paper is to prove that involutions preserving the symplectic form on a Hyperkähler variety of $K3^{[2]}$ -type are all deformation equivalent.

Many papers on automorphisms of Hyperkähler manifolds have appeared in recent years, starting from the foundational work of Nikulin [17] and Mukai [15] and an explicit example of Morrison [14] in the case of K3 surfaces. Then isolated examples of such automorphisms were given by Namikawa [16], by Beauville [3] and later by Kawatani [11] and Amerik [1]. Some further work was done in the case of generalized Kummer varieties by Boissière, Nieper-Wißkirchen and Sarti [5]. Some general work on automorphisms and birational maps was done by Oguiso [21], Boissière [4] and Boissière and Sarti [6], while order 2 automorphisms were analyzed by Beauville [2] and Camere [7]. Before those works on involutions came the work of O'Grady ([19] and [20]) on Double-EPW sextics which are naturally endowed with an antisymplectic involution.

In the following X will always be a Hyperkähler manifold deformation equivalent to the Hilbert square of a K3 surface (i. e. a manifold of $K3^{[2]}$ -type), and σ will denote a holomorphic symplectic form on X . We recall that the integral second cohomology of a Hyperkähler manifold is endowed with the Beauville-Bogomolov integral quadratic form. **Section 2** contains preliminaries on Hyperkähler manifolds and quadratic forms.

Given a bimeromorphic map φ of X we remark that φ^* is defined on $H^2(X, \mathbb{Z})$ since the indeterminacy locus has codimension at least 2. Let $\varphi : X \dashrightarrow X$ be a meromorphic involution, i. e. $\varphi \circ \varphi = Id$. Then

$\varphi^*\sigma = \pm\sigma$. In this paper we are interested mainly in symplectic involutions, i. e. involutions φ such that

$$\varphi^*(\sigma) = \sigma. \quad (1)$$

In **Section 3** we will prove general results concerning finite automorphism groups of manifolds of $K3^{[2]}$ -type. An interesting question is whether these groups are induced by finite automorphism groups on $K3$'s. We formalize this question as follows. Let G be a group acting faithfully on X , we call (X, G) a couple. Two couples (X, G) and (Y, G) are isomorphic if there exists a G -equivariant isomorphism $X \rightarrow Y$. There is a natural notion of deformations of the couple (X, G) :

Definition 1.1. Given a manifold X and a group G acting faithfully on it we call a G -deformation of X (or a deformation of the couple (X, G)) the following data:

- A flat family $\mathcal{X} \rightarrow B$ and a faithful action of the group G on \mathcal{X} inducing fibrewise faithful actions of G .
- A map $\{0\} \rightarrow B$ and a G -equivariant isomorphism $\mathcal{X}_0 \rightarrow X$.

Let G be a cyclic group generated by the automorphism φ of X . We will denote by (X, φ) the couple (X, G) . Let ψ be an automorphism of a $K3$ surface S , we denote $\psi^{[2]}$ the automorphism it induces on $S^{[2]}$.

Definition 1.2. Let X be a Hyperkähler manifold of $K3^{[2]}$ -type endowed with an automorphism of finite order ψ . The couple (X, ψ) is *standard* if there exists a $K3$ surface S and an automorphism ψ' of S such that the couples (X, ψ) and $(S^{[2]}, \psi'^{[2]})$ are deformation equivalent. The couple (X, ψ) is *exotic* if it is not standard.

We remark that not all automorphisms of a manifold of $K3^{[2]}$ -type are standard. In fact there is an example of Namikawa of an exotic automorphism of order 3 of the Fano variety of lines on a particular cubic fourfold (see [16]).

The main result of the present paper is the following:

Theorem 1.3. *Let X be a manifold of $K3^{[2]}$ -type, and let $\varphi \in \text{Aut}(X)$ be a symplectic involution. Then there exists a $K3$ surface S endowed with a symplectic involution ψ such that (X, φ) and $(S^{[2]}, \psi^{[2]})$ are deformation equivalent.*

The above result proves the following conjecture made by Camere:

Conjecture 1.4. [7] *Let X be a Hyperkähler manifold deformation equivalent to the Hilbert square of a $K3$ surface, and let φ be an involution of X preserving the holomorphic symplectic form. Then the fixed locus X^φ does not contain complex tori.*

2 Preliminaries

2.1 Hyperkähler manifolds

This subsection summarizes a few facts about Hyperkähler manifolds, the interested reader can consult [9].

Definition 2.1. Let X be a Kähler manifold, it is called a Hyperkähler manifold if the following hold:

- X is compact.
- X is simply connected.
- $H^{2,0}(X) = \mathbb{C}\sigma_X$, where σ_X is a symplectic form, that is a holomorphic 2-form which is closed and everywhere nondegenerate.

We remark that the isometry class of $H^2(X, \mathbb{Z})$ with the Beauville-Bogomolov form is invariant under smooth deformations.

Example 2.2. Let S be a K3 surface, then $S^{[2]}$ is a Hyperkähler manifold. Furthermore $H^2(S^{[2]}, \mathbb{Z})$ endowed with its Beauville-Bogomolov pairing is isomorphic to the lattice

$$L = U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus (-2). \quad (2)$$

Here U is the hyperbolic lattice, $E_8(-1)$ is the unique unimodular even negative definite lattice of rank 8, (-2) is (\mathbb{Z}, q) with $q(1) = -2$ and \oplus denotes orthogonal direct sum. We remark that there exists a class $v \in H^{1,1}(S^{[2]}, \mathbb{Z})$ of square -2 such that $(v, H^2(S^{[2]}, \mathbb{Z})) = 2\mathbb{Z}$.

Let X be a Hyperkähler manifold of $K3^{[2]}$ -type, thus $H^2(X, \mathbb{Z}) \cong L$. A marking f of X is an isometry $f : H^2(X, \mathbb{Z}) \rightarrow L$. Marked Hyperkähler manifolds are particularly interesting because there exists a moduli space of marked Hyperkähler manifolds and there is a good notion of period map (see [9, Chapter 25]). Let $\mathcal{M}_{K3^{[2]}}$ be the moduli space of marked Hyperkähler manifolds (X, f) deformation equivalent to $K3^{[2]}$, let

$$\Omega = \{\omega \in \mathbb{P}(L \otimes \mathbb{C}), \omega^2 = 0 \quad (\omega, \bar{\omega}) > 0\}$$

be the period domain and let $\mathcal{P} : \mathcal{M}_{K3^{[2]}} \rightarrow \Omega$ be the period map, where $\mathcal{P}(X, f) = f(\sigma_X)$ and σ_X is a symplectic 2-form.

The period map is surjective and it is a local isomorphism, moreover a global Torelli theorem was proven by Verbitsky in our and in several other cases, see [12], [10] and [22]. Whenever this theorem holds it states that two marked Hyperkähler manifolds having the same period are birational. Other useful notions are those of Positive, Kähler and Birational Kähler cones:

Definition 2.3. Let X be a Hyperkähler manifold and let ω be a Kähler class. Let

$$C_X^+ = \{l \in H_{\mathbb{R}}^{1,1}(X), (l, l)_X > 0\}$$

be the set of positive classes in $H_{\mathbb{R}}^{1,1}(X)$ and let the positive cone \mathcal{C}_X be its connected component containing ω .

Let the Kähler cone $\mathcal{K}_X \subset \mathcal{C}_X$ be the set of Kähler classes.

The birational Kähler cone is the union

$$\mathcal{BK}_X = \bigcup_{f : X \dashrightarrow X'} f^* \mathcal{K}_{X'}, \quad (3)$$

where $f : X \dashrightarrow X'$ runs through all birational maps $X \dashrightarrow X'$ from X to another Hyperkähler manifold X' .

There are several results on the structure of these cones, see [9, Section 27 and 28] and [13, Section 9].

Last but not least is a result due to Huybrechts on birational maps of Hyperkähler varieties (see [12, Theorem 3.2]):

Lemma 2.4. *Let (X, f) and (Y, g) be two marked Hyperkähler manifolds such that $\mathcal{P}(X, f) = \mathcal{P}(Y, g)$ and X and Y are birational. Then there exists an effective cycle $\Gamma = Z + \sum_j Y_j$ in $X \times Y$ satisfying the following conditions:*

- Z is the graph of a bimeromorphic map from X to Y .
- The composition $g^{-1} \circ f$ is equal to $\Gamma_* : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$.
- The codimensions of the projections $\pi_1(Y_j)$ and $\pi_2(Y_j)$ are equal.
- If $\pi_i(Y_j)$ has codimension 1 then it is supported by an effective uniruled divisor.

2.2 Lattices and discriminant forms

In this subsection we summarize several notions on lattice theory and we analyze some lattices appearing in the rest of the paper. Most of these results are taken from [18].

First of all let us start with the basic notions of discriminant groups and forms: given an even lattice N with quadratic form q we can consider the group $A_N = N^\vee/N$, which is called discriminant group and whose elements are denoted $[x]$ for $x \in N^\vee$. We denote with $l(A_N)$ the least number of generators of A_N . On A_N there is a well defined quadratic form q_{A_N} taking values inside $\mathbb{Q}/2\mathbb{Z}$, which is called discriminant form. Moreover we call (n_+, n_-) the signature of q and therefore of N as a lattice.

Lemma 2.5. *[18, Corollary 1.13.5] Let S be an even lattice of signature (t_+, t_-) . Then the following hold:*

- If $t_+ > 0$, $t_- > 0$ and $t_+ + t_- > 2 + l(A_S)$ then $S \cong U \oplus T$ for some lattice T .
- If $t_+ > 0$, $t_- > 7$ and $t_+ + t_- > 8 + l(A_S)$ then $S \cong E_8(-1) \oplus T$ for some lattice T .

Lemma 2.6. *[18, Proposition 1.4.1] Let S be an even lattice. There exists a bijection $S' \rightarrow H_{S'}$ between even overlattices of finite index of S and isotropic subgroups of A_S . Moreover the following hold:*

1. $A_{S'} = (H_{S'}^\perp)/H_{S'} \subset A_S$.
2. $q_{A_{S'}} = q_{A_S}|_{A_{S'}}$.

We will often need to analyze primitive embeddings of an even lattice into another one. Let us make some useful remarks whose proofs can also be found in [18]:

Remark 2.7. A primitive embedding of an even lattice S into an even lattice N is equivalent to giving N as an overlattice of $S \oplus S^{\perp_N}$ corresponding to an isotropic subgroup H_S of $A_S \oplus A_{S^{\perp_N}}$. Moreover there exists an isometry $\gamma : p_S(H_S) \rightarrow p_{S^{\perp_N}}(H_S)$ between q_S and $q_{S^{\perp_N}}$ (p_S

denotes the natural projection $A_S \oplus A_{S^\perp_N} \rightarrow A_S$). Notice that this implies $H_S = \Gamma_\gamma(p_S(H_S))$, where Γ_γ is the pushout of γ in $A_S \oplus A_{S^\perp_N}$, i. e.

$$\Gamma_\gamma = \{(a, \gamma(a)), a \in H_S\}. \quad (4)$$

Remark 2.8. Suppose we have an even lattice S with signature (s_+, s_-) and discriminant form $q(A_S)$ primitively embedded into an even lattice N with signature (n_+, n_-) and discriminant form $q(A_N)$. Let K be an even lattice, unique in its genus and such that $O(K) \rightarrow O(q_{A_K})$ is surjective, with signature (k_+, k_-) and discriminant form $-q(A_N)$. It follows from [18] that primitive embeddings of S into N are equivalent to primitive embeddings of $S \oplus K$ into a unimodular lattice T of signature $(n_+ + k_+, n_- + k_-)$, such that both S and K are primitively embedded in T . By **Remark 2.7** an embedding of $S \oplus K$ into a finite overlattice V such that both S and K are primitively embedded into it is equivalent to giving subgroups H_S of A_S and H_N of A_N and an isometry $\gamma : q_{A_S|H_S} \rightarrow -q_{A_N|H_N}$. Finally a primitive embedding of V into T is given by the existence of a lattice with signature (v_-, v_+) and discriminant form $-q_V$.

Keeping the same notation as before we give a converse to these remarks:

Lemma 2.9. [18, Proposition 1.15.1] *Primitive embeddings of S into an even lattice N are determined by the sets $(H_S, H_N, \gamma, K, \gamma_K)$, where K is an even lattice with signature $(n_+ - s_+, n_- - s_-)$ and discriminant form $-\delta$, $\delta \cong (q_{A_S} \oplus -q_{A_N})|_{\Gamma_\gamma^\perp/\Gamma_\gamma}$ and $\gamma_K : q_K \rightarrow (-\delta)$ is an isometry. Moreover two such sets $(H_S, H_N, \gamma, K, \gamma_K)$ and $(H'_S, H'_N, \gamma', K', \gamma'_K)$ determine isometric sublattices if and only if*

- $H_S = \lambda H'_S$, $\lambda \in O(q_S)$.
- There exist $\epsilon \in O(q_{A_N})$ and $\psi \in \text{Isom}(K, K')$ such that $\gamma' = \epsilon \circ \gamma$ and $\bar{\epsilon} \circ \gamma_K = \gamma'_K \circ \bar{\psi}$. Here $\bar{\epsilon}$ and $\bar{\psi}$ are the isometries induced among discriminant groups.

The following is a lemma concerning primitive vectors, we include it here since it is needed in the proof of **Lemma 5.7**:

Lemma 2.10. [8, Lemma 7.5] *Let T be an even lattice such that $T \cong U^2 \oplus N$ for some lattice N , and let $v, w \in T$ be two primitive vectors such that the following hold:*

- $v^2 = w^2$.
- $(v, T) \cong m\mathbb{Z} \cong (w, T)$, where (v, T) is the image of the linear function $(v, -)$ applied to T .
- $[\frac{v}{m}] = [\frac{w}{m}]$ in A_T .

Then there exists an isometry g of T such that $g(v) = w$.

Example 2.11. Let us specialize to the lattices of interest to us. Let $E_8(-2)$ be the lattice E_8 with quadratic form multiplied by -2 . Let L be as in (2) and let

$$\Lambda = U^4 \oplus E_8(-1)^2. \quad (5)$$

$$M = E_8(-2) \oplus U^3 \oplus (-2). \quad (6)$$

The lattice $E_8(-2)$ has discriminant group $(\mathbb{Z}_{/2})^8$ and discriminant form $q_{E_8(-2)}$ given by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

The lattice L satisfies

$$A_L = \mathbb{Z}_{/2}, \quad q_{A_L}(1) = -\frac{1}{2}. \quad (7)$$

Since Λ is unimodular $A_\Lambda = \{0\}$. The lattice (-2) has discriminant group $\mathbb{Z}_{/2}$ and discriminant form q' with $q'(1) = q_{A_{(-2)}}(1) = \frac{1}{2}$. Therefore the lattice M has discriminant form $q_{E_8(-2)} \oplus q'$ over the group $(\mathbb{Z}_{/2})^9$.

Lemma 2.12. *Let $g \in O(L)$, then there exists an embedding $L \subset \Lambda$ and an isometry $\bar{g} \in O(\Lambda)$ such that $\bar{g}|_L = g$ and $\bar{g}|_{L^\perp} = Id$.*

Proof. The isometry g induces an automorphism of the discriminant group A_L . Since $A_L = \mathbb{Z}_{/2}$ this automorphism is the identity. Let $[v/2]$ be a generator of A_L such that $v^2 = -2$. We then have $g([v/2]) = [v/2]$, i. e. $g(v) = v + 2w$. Consider now a lattice of rank 1 generated by an element x of square 2, its discriminant group is still $\mathbb{Z}_{/2}$ and is generated by $[x/2]$ with discriminant form given by $q(x/2) = 1/2$. Notice that $L \oplus \mathbb{Z}x$ has an overlattice isometric to Λ which is generated by L and $\frac{x+v}{2}$.

We now extend g on $L \oplus x$ by imposing $g(x) = x$ and we thus obtain an extension \bar{g} of g to Λ defined as follows:

$$\begin{aligned} \bar{g}(e) &= g(e), \quad \forall e \in L, \\ \bar{g}(x) &= x, \\ \bar{g}\left(\frac{x+v}{2}\right) &= \frac{x+g(v)}{2}. \end{aligned}$$

□

To conclude this section we analyze the behaviour of (-2) vectors inside L and M , since they will play a fundamental role in the proof of **Theorem 1.3**. We will need the following:

Lemma 2.13. *Let (-2) be the usual lattice and let e be one of its generators. Let L and M be as before. Then the following hold:*

- *Up to isometry there is only one primitive embedding $(-2) \hookrightarrow M$ such that $(e, M) = 2\mathbb{Z}$ (i. e. e is 2-divisible). Moreover $e \oplus e^\perp = M$.*
- *Up to isometry there is only one primitive embedding $(-2) \hookrightarrow L$ such that $(e, L) = 2\mathbb{Z}$. Moreover $e \oplus e^\perp = L$.*

Furthermore all other primitive embeddings into M given by $(H_e, H_M, \gamma, K, \gamma_K)$ satisfy the following:

$$\exists s \in A_K, q_{A_K}(s, s) = \pm \frac{1}{2}. \quad (8)$$

Proof. By **Lemma 2.9** we know that the quintuple $(H_e, H_M, \gamma, K, \gamma_K)$ determines primitive embeddings of e inside M and the quintuple $(H_e, H_L, \gamma, K, \gamma_K)$ provides those into L . A direct computation shows that primitive embeddings of e into L are 2-divisible only for the quintuple $(\mathbb{Z}_{/(2)}, A_L, Id, U^3 \oplus E_8(-1)^2, Id)$.

Now let us move on to the case of M . If $H_e = Id$ we have $K \cong U^2 \oplus E_8(-2) \oplus (2) \oplus (-2)$, obviously e is not 2-divisible in this case and this satisfies (8). If $H_e = \mathbb{Z}_{/(2)}$ and $(H_M, A_M^{\perp A_{E_8}}) \neq 0$ we obtain nonetheless condition (8), and again e is not 2-divisible in this embedding since $e \oplus e^{\perp M}$ is properly contained in M with index a multiple of 2. Therefore $(\mathbb{Z}_{/(2)}, A_M^{\perp A_{E_8}}, Id, U^3 \oplus E_8(-2), Id)$ is the only possible case. \square

3 Action of automorphisms on cohomology

In this section we provide a series of useful facts about finite groups acting faithfully on X and provide a generalization of some results contained in [17]. We wish to remark that some among these results are already contained in [3], such as most of **Lemma 3.4** and (10).

Definition 3.1. Let G be a finite group acting faithfully on X , we define the invariant locus $T_G(X)$ inside $H^2(X, \mathbb{Z})$ to be the fixed locus of the induced action of G on cohomology. Moreover we define the co-invariant locus $S_G(X)$ as $T_G(X)^\perp$. The fixed locus of G on X will be denoted as X^G .

Furthermore if a group G acts on a lattice R we define $T_G(R)$ to be the invariant sublattice and $S_G(R) = T_G(R)^\perp$. From now on we keep the same notations as in [17], apart for the following:

Definition 3.2. $T(X)$ is the least integer Hodge structure (i. e. $T(X)$ is a lattice and $T(X) \otimes \mathbb{C}$ is a Hodge structure) such that $\sigma \in T(X) \otimes \mathbb{C}$ and it is called the transcendant lattice. Furthermore let $S(X) = T(X)^\perp$.

Let us remark that $S(X) = H_{\mathbb{Z}}^{1,1}(X)$ if X is projective or generic, in fact in those cases $H_{\mathbb{Z}}^{1,1}(X)^\perp$ is an irreducible Hodge structure and $H_{\mathbb{Z}}^{1,1}(X)^\perp \otimes \mathbb{C}$ contains σ . This definition differs from that given in [17] and in several other papers (where usually $S(X) = H_{\mathbb{Z}}^{1,1}(X)$ and $T(X) = S(X)^\perp$). The same definition can be given for any symplectic manifold.

Example 3.3. An example where our definition differs from the usual one is given by a very general elliptic K3, where we have $H_{\mathbb{Z}}^{1,1}(X) = \mathbb{Z}v$, $v^2 = 0$ and $T(X) \cong v^\perp/v \cong U^2 \oplus E_8(-1)^2$, $S(X) \cong U$.

Let γ_X be the following useful map:

$$\gamma_X : T(X) \rightarrow \mathbb{C}. \quad (9)$$

Given by $\gamma_X(x) = (\sigma, x)_X$, which has kernel $T(X) \cap S(X) = 0$. Moreover we have the following exact sequence for any finite group G of Hodge isometries on $H^2(X, \mathbb{Z})$:

$$1 \rightarrow G_0 \rightarrow G \xrightarrow{\pi} \Gamma_m \rightarrow 1, \quad (10)$$

where $\Gamma_m \subset U(1)$ is a cyclic group of order m . In fact the action of G on $H^{2,0}$ is the action of a finite group on \mathbb{C}^* . We also denoted $G_0 = \ker(G \rightarrow \text{Aut}(H^{2,0}(X)))$. The following result is a generalization of [17, theorem 3.1]:

Lemma 3.4. *Let X be a Hyperkähler 4-fold of $K3^{[2]}$ -type and $G \subset \text{Aut}(X)$, $|G|$ finite. Then the following hold:*

1. $g \in G$ acts trivially on $T(X) \iff g \in G_0$.
2. The representation of Γ_m on $T(X) \otimes \mathbb{Q}$ splits as the direct sum of irreducible representations of the cyclic group Γ_m having maximal rank (i. e. of rank $\phi(m)$).

Proof. 1. Let $g \in G_0$. Let us show that g^* acts trivially on $T(X) \otimes \mathbb{Q}$. We start by considering the kernel of the map $g^* - \text{Id}_{T(X)}$ which is a lattice (and a Hodge substructure) R inside $T(X)$. Hence, by minimality of $T(X)$, $R \otimes \mathbb{Q}$ is either 0 or $R \otimes \mathbb{Q} = T(X) \otimes \mathbb{Q}$. Considering the map (9), since g^* is a Hodge isometry we have

$$\gamma_X(x) = (g^* \sigma, g^* x) = (\sigma, g^* x).$$

Since $g^* \sigma = \sigma$ we have that $g^* x - x \in \ker(\gamma_X) = T(X) \cap S(X) = 0$. Thus R is all of $T(X)$.

To obtain the converse we prove that $g^* \sigma = \lambda \sigma$ with $\lambda \neq 1$ implies that 1 is not an eigenvalue of g^* on $T(X)$. In fact

$$\gamma_X(x) = (g^* \sigma, g^* x) = \lambda \gamma_X(g^* x),$$

i. e. $g^* x \neq x$.

2. The preceding arguments show that every nontrivial element of G/G_0 has no eigenvalue 1 on $T(X)$ and hence also on $T(X) \otimes \mathbb{Q}$, this implies our claim. \square

Let now G be a finite group of automorphisms such that $G = G_0$. Following Nikulin we will call such G an algebraic automorphism group. We want to give some useful generalizations of [17, section 4]:

Lemma 3.5. *Let G be a finite algebraic automorphism group of a fourfold X of $K3^{[2]}$ -type, then the following assertions are true:*

1. $S_G(X)$ is nondegenerate and negative definite.
2. $S_G(X)$ contains no element with square -2 .
3. $T(X) \subset T_G(X)$ and $S_G(X) \subset S(X)$.

4. G acts trivially on $A_{S_G(X)}$.

Proof. The third assertion is an immediate consequence of **Lemma 3.4** because G acts as the identity on σ and therefore on all of $T(X)$.

To prove that $S_G(X)$ and $T_G(X)$ are nondegenerate let $H^2(X, \mathbb{Z}) = \oplus_{\rho} U_{\rho}$ be the decomposition in orthogonal representations of G , where U_{ρ} contains all irreducible representations of G of character ρ inside $H^2(X, \mathbb{Z})$. Obviously $T_G(X) = U_{Id}$ and $S_G(X) = \oplus_{\rho \neq Id} U_{\rho}$, which implies they are orthogonal and of trivial intersection. Hence they are both nondegenerate.

Since G is finite there exists a G -invariant Kähler class ω_G given by $\sum_{g \in G} g\omega$, where ω is any Kähler class on X . Therefore we have:

$$\sigma\mathbb{C} \oplus \overline{\sigma}\mathbb{C} \oplus \omega_G\mathbb{C} \subset T_G(X) \otimes \mathbb{C}.$$

Hence the lattice $S_G(X)$ is negative definite.

To prove the last assertion let us proceed as in **Lemma 2.12**, i. e. let us choose a primitive embedding of $H^2(X, \mathbb{Z})$ in the lattice Λ such that the action of G extends trivially outside the image of $H^2(X, \mathbb{Z})$. Therefore $S_G(X) \cong S_G(\Lambda)$ and $A_{S_G(\Lambda)} \cong A_{T_G(\Lambda)}$, where the isomorphism is G equivariant. G acts trivially on $T_G(\Lambda)$, thus its induced action on $A_{T_G(\Lambda)}$ is trivial. Using the G equivariant isomorphism we have that G acts trivially also on $A_{S_G(\Lambda)} = A_{S_G(X)}$.

Let us prove that there are no -2 vectors inside $S_G(X)$. Assume on the contrary that we have an element $c \in S_G(X)$ such that $(c, c) = -2$. Then by [13, Theorem 1.12] it is known that either $\pm c$ or $\pm 2c$ is represented by an effective divisor D on X . Let $D' = \sum_{g \in G} gD$ which is also an effective divisor on X , but $[D'] \in S_G(X) \cap T_G(X) = \{0\}$. This implies D' is linearly equivalent to 0, which is impossible. \square

Now we can use **Lemma 2.4** to give sufficient conditions for an isometry ψ of L to be induced by a birational map ψ' of some marked Hyperkähler manifold (X, f) such that $f \circ \psi'^* \circ f^{-1} = \psi$. Thus we obtain a generalization of [17, Theorem 4.3]:

Theorem 3.6. *Let G be a finite subgroup of $O(L)$. Suppose that the following hold:*

1. $S_G(L)$ is nondegenerate and negative definite.
2. $S_G(L)$ contains no element with square (-2) .

Then G is induced by a subgroup of $Bir(X)$ for some manifold (X, f) of $K3^{[2]}$ -type.

Proof. By the surjectivity of the period map and by **Lemma 3.5** we can consider a marked $K3^{[2]}$ -type 4-fold (X, f) such that $T(X) \xrightarrow{f} T_G(L)$ is an isomorphism and also $S(X) \xrightarrow{f} S_G(L)$ is.

Let $g \in G$, let us consider the marked varieties (X, f) and $(X, g \circ f)$. They have the same period in Ω and hence by **Lemma 2.4** we have $f^{-1} \circ g \circ f = \Gamma^*$. Here $\Gamma = Z + \sum_j Y_j$ in $X \times X$, where Z is the graph of a bimeromorphic map from X to itself and Y_j 's are cycles with $\text{codim}(\pi_i(Y_j)) \geq 1$.

We will prove that all Y_j 's contained in Γ have $\text{codim}(\pi_i(Y_j)) > 1$, thus implying $\Gamma_* = Z_*$ on $H_{\mathbb{Z}}^2$. We know those of codimension 1 are uniruled and effective, moreover it is known (see [9, Proposition 28.7]) that uniruled divisors cut out the closure of the birational Kähler cone \mathcal{BK}_X , i. e. $(\alpha, D) \geq 0$ for all $\alpha \in \overline{\mathcal{BK}}_X$ and for all uniruled D . We wish to remark that the manifold X we chose has $\overline{\mathcal{BK}}_X = \overline{\mathcal{C}}_X$ by [12, Theorem 9.17] (it contains no -2 divisors). Let $\beta \in \mathcal{C}_X$ be a Kähler class and let $D \in \text{Pic}(X)$ be a uniruled divisor, we can write

$$\beta = \alpha + \gamma, \quad f(\alpha) \in T_G(L) \otimes \mathbb{R}, \quad f(\gamma) \in S_G(L) \otimes \mathbb{R}.$$

Hence $0 < (\beta, D) = (\gamma, D)$ and moreover we have $(f^{-1} \circ g \circ f(\beta), D) = (f^{-1} \circ g \circ f(\gamma), D) = (\gamma, f^{-1} \circ g^{-1} \circ f(D)) \geq 0$ because $f^{-1} \circ g \circ f(\beta) \in \overline{\mathcal{BK}}_X$ and D is uniruled. Here is the contradiction:

$$0 < (\beta, \sum_{h \in G} f^{-1} \circ h \circ f(D)),$$

which implies $0 \neq D' = \sum_{h \in G} hD \in f^{-1}(T_G(L) \cap S_G(L)) = 0$, hence there are no uniruled divisors inside $\text{Pic}(X)$. Moreover we obtain $\Gamma_* = Z_*$, i. e. there exists a bimeromorphic map ψ' of X such that $\psi'^* = f^{-1} \circ g \circ f$ on $H^2(X)$. \square

4 Fixed locus of a Symplectic involution

Our work on symplectic involutions starts with an analysis of the fixed locus of a symplectic involution φ on X . The main result of this section is the following:

Theorem 4.1. *Let X be a Hyperkähler manifold of $K3^{[2]}$ -type with a symplectic involution φ . Then **Conjecture 1.4** holds true, the fixed locus X^φ consists of 28 isolated points and one K3 surface. Moreover the lattice $T_\varphi(X)$ has rank 15.*

Notice that this is what happens in **Example 4.4**. The starting point of our proof will be the following result of Camere:

Proposition 4.2. [7] *Let X be a manifold of $K3^{[2]}$ -type and let $\varphi \in \text{Aut}(X)$ be a symplectic involution. Then $\text{rank}(T_\varphi(X)) \geq 11$. Moreover, unless X^φ contains a complex torus, we have $\text{rank}(T_\varphi(X)) = 15$ and X^φ consists of 28 isolated points and a K3 surface.*

Remark 4.3. **Proposition 4.2** implies that a symplectic involution on X cannot induce the identity map on cohomology. The same result can be proven for automorphisms of manifolds of $K3^{[2]}$ -type of any order.

Example 4.4. Let S be a K3 surface endowed with a symplectic involution ψ . Then the manifold $X = S^{[2]}$ has a symplectic involution $\psi^{[2]}$ fixing 28 points and 1 K3 surface Y . Notice that Y is the minimal resolution of S/ψ .

For further examples the reader can consult [7].

To prove **Theorem 4.1** let us do some preliminary work to analyze small deformations of the couple (X, φ) , i. e. the following: let us choose a small ball U representing $Def(X)$, whose tangent space at the origin is given by $H^1(\mathcal{T}_X)$. The symplectic involution on X extends to an automorphism of the versal deformation family $\mathcal{X} \rightarrow U$ as follows:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{M} & \mathcal{X} \\ \downarrow & & \downarrow \\ U & \xrightarrow{m} & U. \end{array}$$

Here m is the involution on U (which is small enough to have $m(U) = U$), induced by the action of the symplectic involution φ on $H^1(\mathcal{T}_X)$. Moreover m induces an involution of \mathcal{X} which yields fibrewise isomorphisms between \mathcal{X}_t and $\mathcal{X}_{m(t)}$. The differential of m at 0 is given by the action of φ on $H^1(\mathcal{T}_X)$, which is the same as the action on $H^{1,1}(X)$ since the symplectic form σ induces an isomorphism between those two and σ is preserved by the action of φ . On the other hand U^m is smooth, since m is linearizable, and hence

$$\dim(U^m) = \text{rank}(T_\varphi(X)) - 2,$$

which is always positive by **Lemma 3.5**. We wish to obtain a deformation of the couple (X, φ) , hence we need to restrict to U^m to get a fibrewise involution. Therefore we obtain the following diagram:

$$\mathcal{Y} = \begin{array}{ccc} \mathcal{X}|_{U^m} & \xrightarrow{M} & \mathcal{X} \\ \downarrow & & \downarrow \\ U^m & \xrightarrow{m} & U, \end{array} \quad (11)$$

where $\mathcal{Y} \rightarrow U^m$ represents the functor of deformations of the couple (X, φ) , i. e. all small deformations of this couple must embed in $\mathcal{Y} \rightarrow U^m$. The involutions φ_t are given by $M|_{\mathcal{X}_t}$.

It is obvious that this deformation space is "maximal" in some sense. Let us make this more precise using the period map.

Definition 4.5. Given a finite group $G \subset O(L)$ we denote Ω_G the set of points (X, f) in the period domain such that $f(\sigma_X) \in T_G(L)$.

Definition 4.6. Given (X, f) with a group G acting faithfully on it via symplectic bimeromorphic maps, we call the following a maximal family of deformations of (X, G_{Bir})

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathcal{X}_U \\ \downarrow & & \downarrow \\ \{0\} & \xrightarrow{i} & U, \end{array}$$

where the family \mathcal{X} over U is endowed with a fibrewise faithful bimeromorphic action of G and the period map \mathcal{P} , given a compatible marking, sends surjectively a neighbourhood of $0 \in U$ inside a neighbourhood of $\mathcal{P}(X, f) \cap \Omega_G$.

We give the same definition for maximal families (X, G_{Aut}) or (X, G_{Hod}) having G acting as symplectic automorphisms or Hodge isometries on $H^2(X, \mathbb{Z})$ respectively. Notice that the family $\mathcal{Y} \rightarrow U^m$ we stated before is a maximal family for the couple (X, φ) .

Remark 4.7. We remark that the set $\Omega'_G = \bigcup_{v \in T_G(L)} \{x \in \Omega_G : (x, v) = 0\}$ is the union of countable codimension 1 subsets and consists of Hodge structures on marked varieties (X, f) over Ω_G such that the inclusion $f(T(X)) \hookrightarrow T_G(L)$ is proper. Moreover outside this set $T(X)$ is irreducible.

Now we can use this construction to prove the following fact:

Proposition 4.8. *Let (X, φ) be as before and suppose φ fixes at least one complex torus T . Then $T_\varphi(X)$ has rank at most 6.*

Proof. Suppose on the contrary that $T_\varphi(X)$ has rank ≥ 7 . Let us consider small deformations of the couple (X, φ) over a representative U of $Def(X)$ given by

$$\begin{array}{ccc} \mathcal{X}|_{U^m} & \xrightarrow{\Phi} & \mathcal{X} \\ \downarrow & & \downarrow \\ U^m & \xrightarrow{m} & U. \end{array} \quad (12)$$

As shown in (11). We let σ_t be the symplectic form on \mathcal{X}_t . We remark that, by linear algebra, the fixed locus X^φ is smooth and consists only of symplectic varieties since the symplectic form σ restricts to a nonzero symplectic form on all connected components of X^φ . Moreover it is stable for small deformations of the couple (X, φ) , i. e. the fixed locus \mathcal{X}^Φ is a small deformation of the fixed locus X^φ . Therefore we have a well defined map of integral Hodge structures $H^2(\mathcal{X}_t, \mathbb{C})^{\Phi_t} \rightarrow H^2(T_t, \mathbb{C})$ sending a class on $H^2(\mathcal{X}_t)$ to its restriction to T_t , where T_t is a small deformation of T fixed by Φ_t (i. e. is a component of the fibre over t of \mathcal{X}^Φ). Since $\Phi_t(\sigma_t) = \sigma_t$ and $\sigma_t|_{T_t} \neq 0$ this map is not the zero map and, being a map of Hodge structures, its kernel is again a Hodge structure. Given a marking F over \mathcal{X} we have that (\mathcal{X}, F) is a maximal family of deformations of the couple (X, φ) . Let $V = \{\mathcal{P}(\mathcal{X}_t, F_t), t \in U\} \subset \Omega_\varphi$, by **Remark 4.7** there exists $u \in V \setminus \Omega'_\varphi$ and this period corresponds to a marked manifold (\mathcal{X}_t, F_t) such that $T(\mathcal{X}_t) = T_{\Phi_t}(\mathcal{X}_t)$, i. e. this Hodge structure is irreducible. Therefore we have that the map $H^2(\mathcal{X}_t, \mathbb{C})^{\Phi_t} \rightarrow H^2(T_t, \mathbb{C})$ is an injection. But this is absurd if $T_\varphi(X)$ has rank greater than 6 since $H^2(T_t)$ has dimension 6. \square

Proof of Theorem 4.1. By **Proposition 4.2** we have that $rank(T_\varphi(X)) \geq 11$. By **Proposition 4.8** we therefore have that symplectic involutions cannot fix complex tori, hence we have our claim.

5 Deformation equivalence of couples (X, φ)

Having determined X^φ we proceed to compute $S_\varphi(X)$. We will use part of Nikulin's theorems summarized in **Subsection 2.2**, and also the following result concerning invariant and co-invariant lattices of involutions on L and Λ :

Lemma 5.1. *Let $\varphi \in O(L)$ be an involution and let $L \subset \Lambda$ as in*

Lemma 2.12, i. e. $\varphi \in O(L) \subset O(\Lambda)$. *Then the following hold:*

1. *The quotient $\Lambda/(T_\varphi(\Lambda) \oplus S_\varphi(\Lambda))$ is of 2-torsion.*

2. $S_\varphi(L) \cong S_\varphi(\Lambda)$.

Proof. Given an element $t \in \Lambda$ we have $t = \frac{t+\varphi(t)}{2} + \frac{t-\varphi(t)}{2}$ and clearly $t+\varphi(t) \in T_\varphi(\Lambda)$ while $t-\varphi(t) \in S_\varphi(\Lambda)$ so we have $2t \in S_\varphi(\Lambda) \oplus T_\varphi(\Lambda)$. The lattice Λ is generated by L and $\frac{x+v}{2}$ as in **Lemma 2.12**, where $\langle x \rangle = L^\perp$ and v is a 2-divisible vector of square -2 inside L . Moreover a vector $b = a\frac{x+v}{2} + w'$, $w' \in L$ is in $S_\varphi(\Lambda)$ if and only if

$$-a\frac{x+v}{2} - w' = \varphi(a\frac{x+v}{2} + w') = a\frac{x}{2} + \varphi(w') + a\frac{v}{2} + aw.$$

Here $\varphi(v) = v + 2w$. But this can happen only if $a = 0$, i. e. $b \in S_\varphi(L)$ \square

Theorem 5.2. *Let X be a Hyperkähler manifold of $K3^{[2]}$ -type and $\varphi \in \text{Aut}(X)$ a symplectic involution. Then the lattice $S_\varphi(X)$ is isomorphic to $E_8(-2)$ and $T_\varphi(X)$ is isomorphic to $E_8(-2) \oplus U^3 \oplus (-2)$.*

Proof. We know that $S_\varphi(X)$ has rank 8 by **Theorem 4.1** and it equals $S_\varphi(\Lambda)$ by **Lemma 5.1**, therefore the discriminant group $A_{S_\varphi(\Lambda)}$ can be generated by 8 elements and so does its unimodular complement $A_{T_\varphi(\Lambda)}$. This means that we can apply **Lemma 2.5** obtaining $T_\varphi(\Lambda) = U \oplus T'$, which means that we can define an involution of $U^3 \oplus E_8(-1)^2$ having $S_\varphi(\Lambda)$ as the anti-invariant lattice. By **Lemma 3.5** this involution satisfies the conditions of [17, Theorem 4.3] which implies that this involution on $U^3 \oplus E_8(-1)^2$ is induced by a symplectic involution ψ on some $K3$ surface S and hence also $S_\psi(S) \cong S_\varphi(X)$.

Thus, by the work of Morrison on involutions [14], we know $S_\varphi(X) = E_8(-2)$ and $T_\varphi(X)$ is just its orthogonal complement in L , which is easily proven to be $E_8(-2) \oplus U^3 \oplus (-2)$ using **Lemma 2.9**. \square

Corollary 5.3. *Let $M_1, M_2 \subset L$ such that $M_1 \cong M_2 \cong E_8(-2)$. Then there exists $f \in O(L)$ such that $f(M_1) = M_2$.*

Proof. By **Example 2.11** we know the discriminant form and group of $E_8(-2)$. Therefore we can apply **Lemma 2.9**, obtaining that embeddings of $E_8(-2)$ into L are given by quintuples $(H, H', \gamma, K, \gamma_K)$. Moreover two such embeddings $(H, H', \gamma, K, \gamma_K)$ and $(N, N', \gamma', K', \gamma'_{K'})$ are conjugate if and only if we have H conjugate to N through an automorphism of $(\mathbb{Z}/(2))^8$ sending γ into γ' . In our case the computations are particularly simple: due to the values of $q_{E_8(-2)}$ (all elements have square 0 or 1) and q' (all nonzero elements have square $\frac{1}{2}$) the only possible choices of H and H' are given by the one element group and so we obtain our claim.

Moreover this implies that we can always choose a marking of (X, φ) such that the induced action of φ on L is given by leaving $(-2) \oplus U^3$ invariant and exchanging the two remaining $E_8(-1)$, so that S_φ is given by the differences $a - \varphi(a)$ for $a \in E_8(-1)$. \square

Now we can proceed to the proof of **Theorem 1.3**, i. e. that all couples (X, φ) where φ is a symplectic involution are standard. Let us start by defining a space containing any (X, φ) :

Definition 5.4. Let \mathcal{M}_2 be the subset of $\mathcal{M}_{K3^{[2]}}$ given by the marked manifolds (X, f) such that:

$$\exists V \cong E_8(-2), V \subset L : V \subset f(H_{\mathbb{Z}}^{1,1}(X)).$$

Proposition 5.5. *Let (X, φ) be a couple consisting of a $K3^{[2]}$ -type manifold and a symplectic involution of X . Then \mathcal{M}_2 contains all couples (X, f) for any marking f . Moreover the generic point of \mathcal{M}_2 corresponds to a marked manifold (Y, g) having a bimeromorphic involution.*

Proof. \mathcal{M}_2 is locally given by 8 linearly independent conditions on the image of the period map, i. e. $\mathcal{P}(X, f) \perp a$ with a ranging through a set of generators for a lattice of type $E_8(-2) \subset L$. Due to **Corollary 5.3** we assume that the marking is fixed. Given such an (X, f) we can define an involution φ inside its cohomology by imposing $f(S_{\varphi}(X)) = E_8(-2) \subset L$. Since this is a maximal family of Hodge involutions, the generic element (Y, g) of this space has $\text{Pic}(Y) = E_8(-2)$ by **Remark 4.7** and we know by **Theorem 3.6** that φ extends to a birational involution on Y . Finally a couple (X, φ) endowed with a marking f satisfies the condition $E_8(-2) \subset f(\text{Pic}(X))$ by **Theorem 5.2** and is thus inside this space. \square

Definition 5.6. Let $\Omega_2 = \mathcal{P}(\mathcal{M}_2)$ and furthermore let $\Omega_{v,2}$ denote the set of $\omega \in \Omega_2$ such that $(v, \omega) = 0$.

Let M be as in (6), there is a sublattice M_0 of L isomorphic to M given by $f(T_{\varphi}(S^{[2]}))$, where $(S^{[2]}, f)$ is a marked Hyperkähler manifold and φ is a symplectic involution on it. Moreover, by **Corollary 5.3**, all such lattices are conjugate through an isometry of L , hence without loss of generality we fix $M_0 \subset L$, $M \cong M_0$ and we can impose

$$\mathcal{P}(X, f) \in \mathbb{P}(M_0 \otimes \mathbb{C})$$

for all couples (X, φ) and an appropriate marking f .

Lemma 5.7. *Let $0 \neq w \in M$ be a primitive isotropic vector, then there exist a sublattice $w \in T \subset M$ and a (-2) vector p such that:*

- p is 2-divisible in M ,
- $q_{M|T}$ is nondegenerate,
- $R := T^{\perp_M} \cong U \oplus \langle p \rangle \oplus R'$ for some lattice R' .

Proof. Since $M = U^2 \oplus (U \oplus E_8(-2) \oplus (-2))$ we can apply **Lemma 2.10**. Therefore we can analyze up to isometry all isotropic vectors inside M knowing only their divisibility m (i. e. $(w, M) = m\mathbb{Z}$) and their image $[\frac{w}{m}]$ in A_M . Let us give a basis of M as follows:

$$\{e_1, f_1, e_2, f_2, e_3, f_3, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, t\}, \quad (13)$$

where $\{e_i, f_i\}$ is a standard basis of U , $\{a_1, \dots, a_8\}$ is a standard basis of $E_8(-2)$ and t is a generator of the lattice (-2) .

The first key remark is that since A_M is of 2-torsion m can either be 1 or 2. Therefore if $m = 1$ we have that $\frac{w}{m}$ lies in M , which implies $[\frac{w}{m}] = 0$ in A_M . Thus by **Lemma 2.10** there exists an isometry g of M sending w to e_1 . To obtain our claim we let $T = g^{-1}(\langle e_1, f_1 \rangle)$, $p = g^{-1}(t)$ and

$R = g^{-1}(< e_2, f_2, e_3, f_3, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, t >)$.
If $m = 2$ we have that $\frac{w}{2}$ is a square zero element of M^\vee , i. e. $[\frac{w}{2}]$ has square zero in A_M . Looking at **Example 2.11** it is easy to see that square zero elements must lie in $A_{E_8(-2)} \subset A_M$ and they are given by $[\frac{v}{2}]$ where v is a primitive vector of square $c \equiv 0 \pmod{8}$ inside $E_8(-2)$. Therefore by **Lemma 2.10** there exists an isometry g of M sending w to $r = 2e_1 + \frac{c}{4}f_1 + v$. Thus we set $T = g^{-1}(< r, f_1 >)$, $p = g^{-1}(t)$, $K = v^{\perp_{E_8(-2)}}$ and $R = g^{-1}(< e_2, f_2, e_3, f_3, K, t >)$. \square

Lemma 5.8. *Let $0 \neq w_0 \in M_0$ be a primitive vector of square 0 and let $k \in \mathbb{Z}$.*

There exists a sequence $\{w_n\}$ of non-zero primitive vectors of norm $2k$ such that $[w_n]$ converges to $[w_0]$ in $\mathbb{P}(M_0 \otimes \mathbb{R})$. Moreover if k is odd we may assume that for all n :

$$(w_n, L) = 2\mathbb{Z}.$$

Furthermore if $k = 0$ we may assume that there exists an element q of square -2 and divisibility 2 such that $w_n \perp q$.

Proof. We keep the same notation as in **Lemma 5.7** and we fix an isometry $\eta : M_0 \rightarrow M$.

Let $w = \eta(w_0)$, since it satisfies the hypothesis of **Lemma 5.7** we have a lattice $p \oplus U$ orthogonal to w , where p is a 2-divisible (-2) vector. Let e and f be two standard generators of such U . The sequence $\{[\eta^{-1}(nw + e + kf)]\}_n$ converges to $[w_0]$ in $\mathbb{P}(M_0 \otimes \mathbb{R})$ and consists of primitive vectors of square $2k$.

If k is odd the sequence $\{[\eta^{-1}(2nw + p + 2e + (k+1)f)]\}_n$ converges to $[w_0]$ in $\mathbb{P}(M_0 \otimes \mathbb{R})$, we need only to prove that it is composed by 2-divisible vectors. Obviously this is equivalent to proving that $\eta^{-1}(p)$ is 2-divisible in L , i. e. to proving that $\eta^{-1}(p) \oplus \eta^{-1}(p)^{\perp_L} = L$. We know that $p^{\perp_M} \cong U^3 \oplus E_8(-2)$ hence $\eta^{-1}(p)^{\perp_L}$ is an overlattice of $U^3 \oplus E_8(-2)^2$ which, by **Lemma 2.13**, implies $\eta^{-1}(p)$ is 2-divisible in L . To obtain our last claim we let $q = \eta^{-1}(p)$, as we just proved it is 2-divisible, of square -2 and it is orthogonal to w_n . \square

Definition 5.9. Let $\mathcal{P}_{exc} = \{f \in M_0 : f^2 = -2, (f, L) = 2\mathbb{Z}\}$ be the set of exceptional primitive classes inside M_0 .

Notice that $\mathcal{P}^{-1}(v)$ contains the Hilbert square of a $K3$ surface for all $v \in \mathcal{P}_{exc}$.

Lemma 5.10. $\cup_{v \in \mathcal{P}_{exc}} \Omega_{v,2}$ is dense in Ω_2 .

Proof. It is enough to prove that $\cup_{v \in \mathcal{P}_{exc}} \Omega_{v,2}$ is dense in $\Omega_2 \cap \mathbb{P}(M_0 \otimes \mathbb{C})$ by **Corollary 5.3**.

Let Q_{M_0} be the subset of isotropic vectors inside $\mathbb{P}(M_0 \otimes \mathbb{C})$. Let $Q_{M_0}(\mathbb{R})$ and $Q_{M_0}(\mathbb{Q})$ be the subsets of isotropic vectors spanned by real (respectively rational) isotropic vectors. Let ω be in $\Omega_2 \cap \mathbb{P}(M_0 \otimes \mathbb{C})$, we have $\omega^{\perp_{M_0}} \cap Q_{M_0}(\mathbb{R}) = (\alpha\omega + \overline{\omega\sigma})^{\perp_{M_0}} \cap Q_{M_0}(\mathbb{R})$.

But since $(\alpha\omega + \overline{\alpha\omega})^{\perp_{M_0}}$ has signature $(1,12)$ we have that $\exists u \in Q_{M_0}(\mathbb{R}) \cap (\alpha\omega + \overline{\alpha\omega})^{\perp_{M_0}}$. Since $Q_{M_0}(\mathbb{Q})$ is non-empty it is dense inside $Q_{M_0}(\mathbb{R})$, therefore $\exists \{v_n\}$ such that $[v_n] \rightarrow [u]$ in $\mathbb{P}(M_0 \otimes \mathbb{C})$, where the v_n are

primitive isotropic vectors inside M_0 . Thus we can apply **Lemma 5.8** to find a sequence $\{w_n\}$ of elements of \mathcal{P}_{exc} and a sequence $\{v'_n\}$ of isotropic primitive vectors such that $[v'_n] \rightarrow [u]$ and $w_n \perp v'_n$. \square

Proof of Theorem 1.3. Let f be a marking of X such that $\mathcal{P}(X, f) \subset \mathbb{P}(M_0 \otimes \mathbb{C})$ and $f(S_\varphi(X)) \perp M_0$. Moreover let $\mathcal{X} \rightarrow U$ be a maximal family of deformations of the couple (X, φ) as in (11) and let F be a marking of \mathcal{X} compatible with f such that $V = \{\mathcal{P}(\mathcal{X}_t, F_t), t \in U\}$ is a small neighbourhood of $\mathcal{P}(X, f)$. By **Lemma 5.10** there exist a point $v \in V$ and a 2-divisible primitive vector e of square (-2) such that $v \perp e$. Since the global Torelli theorem holds we can use **Lemma 2.4** on the manifold \mathcal{X}_u such that $\mathcal{P}(\mathcal{X}_u, F_u) = v$. This gives that \mathcal{X}_u is bimeromorphic to the Hilbert square of a certain K3 surface S .

Thus we get a bimeromorphic involution φ on $S^{[2]}$ such that $S_\varphi(S^{[2]}) \subset \text{Pic}(S) \subset \text{Pic}(S^{[2]})$, where

$$\text{Pic}(S) = \{t \in \text{Pic}(S^{[2]}), e \perp t\}.$$

By [17, Theorems 4.3 and 4.7] we have a symplectic involution ψ on S given by the action of φ on $e^\perp \cong H^2(S, \mathbb{Z})$ which induces an involution $\psi^{[2]}$ on $S^{[2]}$. Furthermore the birational map $\psi^{[2]} \circ \varphi$ induces the identity on $H^2(S^{[2]}, \mathbb{Z})$, therefore it is biregular (sends any Kähler class into itself), and it is also the identity (see **Remark 4.3**). This means $\varphi = \psi^{[2]}$, which implies our claim.

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